INDIAN STATISTICAL INSTITUTE Probability Theory II: B. Math (Hons.) I Semester II, Academic Year 2023-24 Final Exam

Date: 29/04/2024 Full Marks: 50 Duration: 3 hours

- Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.
- You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.
- 1. Suppose X and Y independent and identically distributed standard normal random variables. Define Z := X/Y.
 - (a) (7+1=8 marks) Compute a probability density function of Z. What distribution does Z follow?
 - (b) (7 marks) Does Z have finite mean? Please justify your answer.
- 2. (8+2=10 marks) Suppose an unbiased die is thrown *n* times *independently* of each other. For each $k \in \{1, 2, \ldots, 6\}$, let X_k denote the number of k's that are obtained out of these *n* throws. Calculate the correlation coefficient between X_1 and X_6 . Please justify the sign of this correlation coefficient heuristically.
- 3. Suppose (X, Y) is a uniformly chosen point from the region

 $\Delta := \{ (x, y) \in \mathbb{R}^2 : |x| < 1 - y, \ y > 0 \}.$

- (a) (1+1 = 2 marks) What are the marginal ranges of the random variables X and Y?
- (b) (6 marks) For each $x \in Range(X)$, calculate a conditional probability density function of Y given X = x.
- (c) (6 marks) For each $y \in Range(Y)$, calculate a conditional probability density function of X given Y = y.
- (d) (2+3=5 marks) Using (c), compute E(X|Y=y) and Var(X|Y=y) for each $y \in Range(Y)$.
- (e) (2 + 4 = 6 marks) Using (d), verify (by direct computations) that E(X) = E[E(X|Y)] and Var(X) = E[Var(X|Y)] + Var[E(X|Y)].